Near-Equal Temperament

Project N-ET : Part 1 of 3

Occupation: Music and Math
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Introduction

In acoustics the accuracy of Equal Temperament (ET) always been compromised, hence the quality of the sound. To solve this problem, a new theory has been developed call Near-Equal Temperament (or N-ET). The kernel of N-ET is a stack of just intervals – 7 perfect fifths (3/2) and 1 major third (5/4). The process of manual tuning is time consuming but the mathematical precision of the true Equal Temperament could be achieved with unconditional accuracy, something that has never been done before. To make this article suitable for everyone, following analysis goes through a basic music theory, math and sound.

Sound Perception (Pitch)

The ability to hear the difference between two frequencies is not linear. For example, take the standard frequency of A4 = 440Hz and increase it with 50%. The result is 660Hz but the pitch does not increase with 50%. We hear the difference between 440Hz and 660Hz higher than expected.

Sound perception is an exponential number of any referent frequency. Using same example, to increase the pitch with 50% ($\frac{1}{2}$ or 0.5), it must be done this way:

$$F = 440 \times 2^{0.5} = 440 \times 2^{\frac{1}{2}} = 440 \times \sqrt{2} = 440 \times 1.4142135623731 \approx 622\ Hz$$

In the well-known Equal Temperament (or ET), the frequencies are calculated by this conception in order to get equally arranged (tempered) tones as per our perception. Following formula defines the ratios of the 12-tone music scale.

$$Ratio_n = \sqrt[12]{2}^{\,n} = 2^{\frac{n}{12}} = 2^{\frac{0}{12}} \dots 2^{\frac{11}{12}}$$

Linear Representation

The intervals in a music scale are measured in cents. Its logarithmic property ensures better understanding how the pitch changes for particular frequency range. As already said, sound perception is not linear so this measurement system shows linearly how the pitch changes. Here are different variations of the formula.

$$Pitch = 1200 \times \frac{ln\left(\frac{frequency_2}{frequency_1}\right)}{ln(2)} = 1200 \times \frac{ln\left(\frac{Ratio_n}{Ratio_m}\right)}{ln(2)} = 1200 \times \frac{ln(Ratio)}{ln(2)}$$

The tones in the Equal Temperament are equally arranged with 100 cents in between. Here is an example with semitone.

$$Pitch_1 = 1200 \times \frac{ln(Ratio_1)}{ln(2)} = 1200 \times \frac{ln(\sqrt[12]{2})}{ln(2)} = 1200 \times \frac{0.0577622650466622}{0.693147180559945} = 100\ cents$$

Another example with a whole tone, between 2 ratios.

$$Pitch_2 = 1200 \times \frac{ln\left(\frac{Ratio_7}{Ratio_5}\right)}{ln(2)} = 1200 \times \frac{ln\left(\frac{\sqrt[12]{2}^7}{\sqrt[12]{2}^5}\right)}{ln(2)} = 200\ cents$$

Just Noticeable Difference (JND)

JND is a general term in psychology to determine what's the smallest unit of which we can notice a difference in light, colour, smell, pain, sound, etc. In music the JND is expressed in cents. JND is not limited to particular frequencies. It always varies for different acoustic instruments. Primary depends on timbre and number of overtones they can produce. According to below research (Springer Handbook of Speech Processing), our best perception in noticing a difference is 1Hz in the range between 80Hz - 500Hz for complex sounds.

Springer Handbook of Speech Processing:
https://books.google.com/books?id=Slg10ekZBkAC&pg=PA65

Consider this statement is true. Then find what is the smallest deviation we can hear for different ranges.

$$JND = 1200 \times \frac{ln\left(\frac{frequency_a}{frequency_b}\right)}{ln(2)} = 1200 \times \frac{ln\left(\frac{500}{499}\right)}{ln(2)} = 3.46593518979162 \approx 3.5\ cents$$

$$JND_2 = 1200 \times \frac{ln\left(\frac{frequency_c}{frequency_d}\right)}{ln(2)} = 1200 \times \frac{ln\left(\frac{81}{80}\right)}{ln(2)} = 21.5063\ cents$$

In other words, if the pitch changes with less than **3.5 cents**, we won't be able to notice a difference.

Tuning Systems Comparison

Below formulas compare the perfect fifths (P5) of Equal Temperament and Just intonation. P5 ratio in Equal Temperament is expressed by $2^{\frac{7}{12}}$. In Just intonation it is $\frac{3}{2}$. Here are their differences in cents.

$$Pitch_{ET} = 1200 \times \frac{ln(Ratio_{P5})}{ln(2)} = 1200 \times \frac{ln\left(\sqrt[12]{2}^7\right)}{ln(2)} = 700\ cents$$

$$Pitch_{JI} = 1200 \times \frac{ln(Ratio_{P5})}{ln(2)} = 1200 \times \frac{ln\left(\frac{3}{2}\right)}{ln(2)} = 701.955\ cents$$

$$Difference = Pitch_{JI} - Pitch_{ET} = 701.955 - 700 \approx 1.955\ cents$$

Acoustic Tuning

When it comes to acoustic tuning, musicians rely only on their trained hearing because digital devices are not precise enough to get the right pitch. Usually they do it by a stack of perfect fifths (P5) or perfect fourths (P4). The ratio of P5 in Equal Temperament is:

$$Ratio_{P5} = \sqrt[12]{2}^{\,7} = 1.49830707687668$$

But how professionals are supposed to catch this completely incomprehensible ratio? Usually, they begin from the just interval 3/2. But it is not accurate enough. So they have to slightly lower the tune of P5 by 1.955 cents. How? – The way they feel it. Then the mathematical precision of Equal Temperament always been compromised, therefore its quality.

Some musicians claim they can hear a difference about 1 cent for grand pianos. This is not officially confirmed, neither by a scientific proof. Bear in mind that errors in manual tuning are overlaid. If 1 cent error is the best precision ever achieved, then in the 12-tone scale the final deviation will be 11 cents (11 tones × 1 cent error). It's not 12 because the 1st tone is the kernel (A4=440Hz). However, this error is beyond acceptable and the harmony is completely ruined.

What about if we stick to scientific prove that JND is 3.5 cents? Then the final deviation will be 11 × 3.5 = 38.5 cents. Do not be surprised why Equal Temperament has been accused for destroying the pure harmony.

No matter how experienced the musicians are, the physiological limits do not allow any human being to be precise enough. Did we ever hear the true Equal Temperament or always been a sort of ET? If digital devices are not precise enough neither the professionals, does it mean we are still compromising the quality of this tuning system since centuries?

So far, more questions arise than answers. The good news is there is a solution of this problem. Further chapters and supplementary documents explain everything in details.

Ratios Relationship

Within an octave, the most harmonic cord is major triad. Let's examine its intervals - major third (M3) and perfect fifth (P5). In Just Intonation they are presented by small whole number.

$$Ratio_{M3} = \frac{5}{4}$$

$$Ratio_{P5} = \frac{3}{2}$$

In manual tuning these intervals can be caught with an unconditional accuracy. Any mistake would affect the sound quality because both soundwaves would be dephased or out of sync. That will be quite obvious for the professionals. In Just Intonation the error in manual tuning is practically zero.

Here are the deviations of major third (M3) and perfect fifth (P5) compared to Equal Temperament.

$$Error_{M3} = 1200 \times \frac{ln\left(\frac{5}{4}\right)}{ln(2)} - 400 = 386.313713864835 - 400 = -13.686286135165\ cents$$

$$Error_{P5} = 1200 \times \frac{ln\left(\frac{3}{2}\right)}{ln(2)} - 700 = 701.955000865387 - 700 = +1.95500086538698\ cents$$

For a moment just ignore the negative result of M3. See both numbers closely. They give an impression that the error of P5 is 7 times M3.

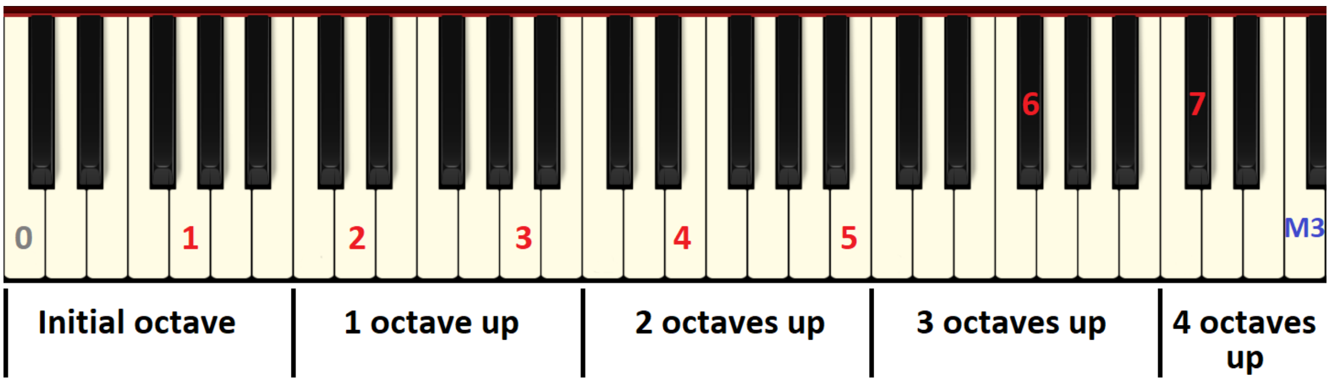
$$\frac{Error_{M3}}{Error_{P5}} = \frac{13.686286135165}{1.95500086538698} = 7.00065477078747$$

The relationship between 7 perfect fifths and 1 major third is almost 0 cents error.

$$Error_{M3} + 7 \times Error_{P5} = -13.686286135165 + 7 \times 1.95500086538698 =$$

$$= -13.686286135165 + 13.6850060577089 = -0.0012800774560997 \ \ cents$$

Therefore, a stack of 7 perfect fifths and 1 major third in Just Intonation should give us a tone with zero cents error. Use a piano keyboard to find out which tone is that. In below picture, red digits are perfect fifths and M3 is a major third.



The result is perfect fourth (P4) and it is 4 octaves higher from the initial step. Then if we create 7 perfect fifths (ratio $\frac{3}{2}$) and 1 major third (ratio $\frac{5}{4}$), eventually we have to shift down the octave 4 times (2^4). The formula of this statement is:

$$Stack = \left(\frac{3}{2}\right)^7 \times \frac{5}{4} \times \frac{1}{2^4} = \left(\frac{3}{2}\right)^7 \times \frac{5}{64} = \frac{2187 \ \times 5}{128 \times 64} = \frac{10935}{8192} = 1.3348388671875$$

For a reference here is the ratio of M3 in Equal Temperament.

$$EtRatio_{P4} = \sqrt[12]{2}^{\,5} = 1.33483985417003$$

Both results are very close. But to conclude whether above stack gives an equally tempered tone, it must be expressed in cents as well.

$$Stack_{cents} = 1200 \times \frac{ln(Stack)}{ln(2)} = 1200 \times \frac{ln(1.3348388671875)}{ln(2)} = 499.998719922547 \ cents$$

$$Stack_{error} \ = Stack_{cents} - 500 = 499.998719922547 - 500 = -0.00128007745297509 \ cents$$

It is the same error found by previous formula ($Error_{M3} + 7 \times Error_{P5}$). This is the error for a single tone tuning. Same procedure must be repeated for the keys that left. To do this, same stack must be built on the top of previous one. This process repeats 11 times because the standard pitch of A4 is the kernel. Eventually the worst error will be 11 times initial error. Remember that the errors in cents are linear, not exponential.

$$FinalError = Stack_{error} \ \times 11 = -0.00128007745297509 \ \times 11$$
$$FinalError = -0.014080851982726 \ cents$$

Near-Equal Intervals

Introducing new terms: the *Stack* variable from previous topic will be called a near-equal interval and will be a part of Near-Equal Temperament. Why near-equal? Because it's close more than enough to Equal Temperament but it's not perfect. The perfect Equal Temperament does not exist in acoustics. It is achievable only for electronic instruments which generate the sound in real time and are sample free.

To find all near-equal ratios (*NER*) in the scale, repeat the $Stack$ 11 times. Build same stack over the previous one. The formula looks like this:

$$NER_n = (Stack)^n = \left(\frac{10935}{8192}\right)^n = \left(\frac{10935}{8192}\right)^0 \dots \left(\frac{10935}{8192}\right)^{11}$$

If the result of any NER_n is greater than 2, divide it by 2. That means the ratio went 1 octave up and must be shifted down.

Below table is for demonstration purpose to show that the ratios of Near-Equal Temperament and Equal Temperament are precise till 4th digit after the decimal point. The worst error in cents is marked with red. The same as $FinalError$ in previous topic.

	Equal Temperament	**Near-Equal Temperament**	
No.	Ratio	Ratio	Error in cents
1	**1**	**1**	0
2	**1.0594**630943593	**1.0594**591775223	-0.00640038726641023
3	**1.12246**204830937	**1.12245**374883624	-0.0128007745327352
4	**1.18920**711500272	**1.18920**447710025	-0.00384023235977793
5	**1.25992**104989487	**1.25991**359721447	-0.0102406196260745
6	**1.33483**985417003	**1.33483**88671875	-0.00128007745325931
7	**1.4142**135623731	**1.4142**0728835527	-0.00768046471955586
8	**1.49830**707687668	**1.49829**489056693	**-0.0140808519857956**
9	**1.5874**010519682	**1.5873**963570668	-0.00512030981280986
10	**1.68179**283050743	**1.68178**163885989	-0.0115206970793906
11	**1.78179**743628068	**1.78179**480135441	-0.00256015490651862
12	**1.88774**862536339	**1.88773**885475646	-0.00896054217287201

Conclusion

Near-Equal Temperament mathematically proved that the errors are practically insignificant for any instrument. To show how accurate is, divide JND by $FinalError$.

$$\frac{JND}{FinalError} = \frac{3.46593518979162}{0.014080851982726} \approx 246\ times\ below\ JND$$

Such accuracy has never been achieved in acoustics. It is possible because working with small ratios $\frac{5}{4}$ and $\frac{3}{2}$ should makes the "near-equal intervals" error free.

Manual Tuning

Take a look again the keyboard picture from chapter Ratios Relationship. Too many steps should be done in order to get a single equally tempered tone. Some people may consider this method as an overhead. There are 2 methods for manual tuning described in supplementary documents – **Forward Stack** and **Fast Inverse Stack**.

It is highly recommended to master the Forward Stack first, before jumping into its improved version called Fast Inverse Stack.

Forward Stack

The required just intervals for this process are $\frac{3}{2}$ and $\frac{5}{4}$. The procedure is for educational purpose to understand how a stack of 7 P5s and 1 M3 works.

Remember the picture from chapter **Ratios Relationship**. The practical implementation of this conception is shown in next picture. The difference is that an octave down shifting is required when we go to a higher octave.

Keys marked with numbers represent all mandatory steps for a single stack. Red digits are perfect fifths ($\frac{3}{2}$) ; blue one is a major third ($\frac{5}{4}$). The starting point is marked with 0. Digits with prime symbol (‘) mark keys with an octave down. From 0 to 7 are all perfect fifths. M3 is a major third tuned from key 7’.

Follow the same process for the other 10 tones that left. Does not matter which tone you are going to start from. The given example is with C4 for better understanding. Do not start building next stack from the tones in referent octave! If you do, you will detune the already tuned keys. Following the logic of same picture, the next stack must start from **M3**, not from M3’.

Read the full manual from supplementary document **Forward Stack**.

Fast Inverse Stack

It is similar to previous one. The required just intervals are: $\frac{3}{2}$; $\frac{3}{4}$; $\frac{5}{4}$.

From the previous picture, probably you already noticed that the intervals between keys denoted with digits 2’-1; 4’-3; 6’-5; 7’-6’ is perfect fourth (P4). Since they have been created in a forward stack, these intervals have the meaning of inversed P4s. The ratio of P4 is $\frac{4}{3}$, so the inversed P4 is $\frac{3}{4}$.

To prove this statement, take an example with a tuning range between C3 and B5 as shown above.

$$D5 = G4 \times \frac{3}{2}$$

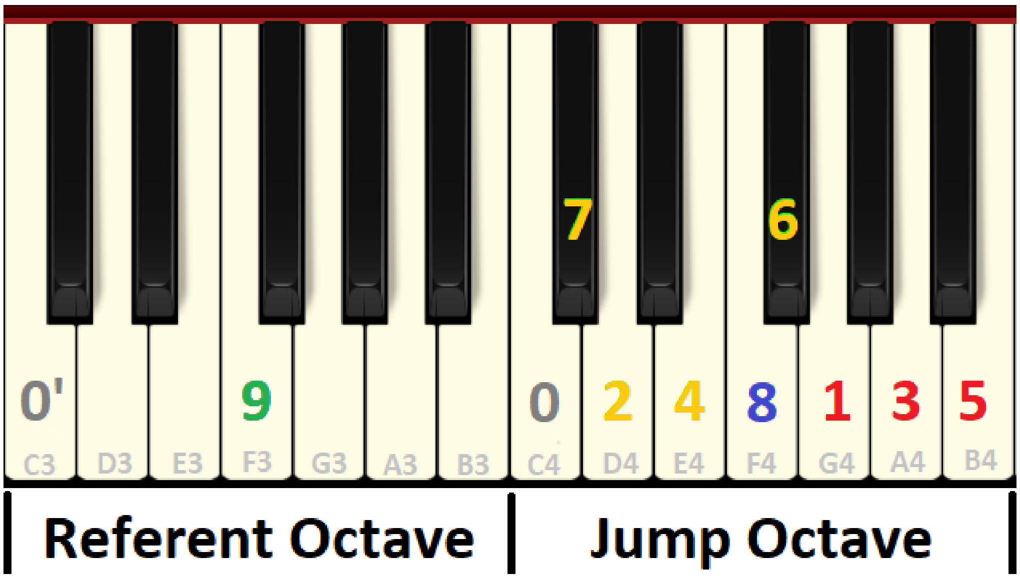
$$D4 = \frac{D5}{2} = D5 \times \frac{1}{2} = G4 \times \frac{3}{2} \times \frac{1}{2} = G4 \times \frac{3}{4}$$

This eliminates the need to use 2 jump octaves. Only 1 referent and 1 jump octaves are required. Tune D4 as a perfect fourth in terms of G4 (inversed perfect fourth). Do not change the tune of G4!

Do not change the tune of G4! It must stay in ratio $\frac{4}{3}$ from D4.

$$G4 = D4 \times \frac{4}{3}$$

In next picture all steps are marked with numbers. The logic is same as described in chapter Forward Stack. Red digits are perfect fifths ($\frac{3}{2}$) ; blue one is a major third ($\frac{5}{4}$). The starting point is marked with 0. Green one is for an octave down.



The only new thing here are the yellow steps. Bear in mind they have the meaning of **inversed perfect fourths**, ratio $\frac{3}{4}$.

Read the full manual from supplementary document “N-ET - Fast Inverse Stack”.